Int. J. Solids Structures Vol. 23, No. 5, pp. 621-630, 1987
Printed in Great Britain.

0020-7683/87 $3.00+.00

A GENERAL FAILURE CRITERION FOR PLAIN CONCRETE

L. F. BOSWELL
Department of Civil Engineering, The City University, Northampton Square, London EC1V 0HB, U.K.

and

Z. CHEN
Tianjin Design Institute, Ministry of Water Conservancy and Electric Power, Tianjin, People's Republic of China

(*Received* 22 *May* 1985; *in revised form* 25 *April* 1986)

Abstract—A general three-dimensional failure criterion for plain concrete is proposed. The criterion has been formulated in terms of three normalized stress invariants and includes material parameters which may be obtained from experimental data. It has been shown that simple failure criteria such as the von Mises model, the Drucker–Prager model and the Bresler–Pister model are special cases of that proposed. The application of the criterion has been demonstrated by comparison with available experimental results. From the results of this comparison it can be seen that the criterion may be applied throughout the stress range from tensile stress to high compressive stress.

INTRODUCTION

There are many examples in the design of concrete structures when a satisfactory explanation of the ultimate strength behaviour can only be achieved if the concrete is considered to be subjected to a three-dimensional stress state.

In recent years a great deal of work has been undertaken, the results of which have been used to propose different failure criteria for concrete[1–14]. Reviews of this work have been presented by Chen[15], Wastiels[16] and Argyris *et al.*[17].

The advent of the use of digital computers for structural analysis has enabled the complex nature of concrete to be represented by numerical models[9–11, 14]. In particular, the five parameter model suggested by Willam and Warnke[10], which is applicable to the triaxial stress state, appears to be the most general model for concrete. This model involves all stress invariants and has a non-circular section in the deviatoric plane, which changes from nearly triangular to nearly circular, with increasing hydrostatic pressure.

A general three-dimensional failure criterion for concrete is proposed in this paper. A particular feature of the formulation is that the eight material parameters are obtained explicitly from data for maximum loads at various stress ratios. The criterion may be considered to be an alternative formulation to those which have been proposed previously[9–11, 14]. It may be used as an ultimate strength surface, when the concrete is treated as an elastic–fracture material, or as a yield surface in the case of elastic–perfectly plastic material.

FAILURE CRITERION

The proposed criterion for failure in concrete is expressed in terms of functions of the stress invariants $\bar{\sigma}_m$, $\bar{r}$ and θ as

$$\left.\begin{aligned} R &= \frac{\bar{r}}{\bar{r}_t} = \{3/[3 - 4(1 - n^2)\sin^2\theta]\}^{1/2} \\ n &= \frac{\bar{r}_t}{\bar{r}_c} \\ \bar{r}_t &= (a_t + b_t\bar{\sigma}_m + c_t\bar{\sigma}_m^2)^{1/2} - d_t \\ \bar{r}_c &= (a_c + b_c\bar{\sigma}_m + c_c\bar{\sigma}_m^2)^{1/2} - d_c \end{aligned}\right\} \tag{1}$$

in which

$$\bar{\sigma}_m = \bar{I}_1/3$$

is the normalized mean normal stress

$$\bar{r} = \sqrt{(2\bar{J}_2)}$$

is the normalized deviatoric stress magnitude, and (2)

$$\cos 3\theta = \frac{3\sqrt{3}}{2}\bar{J}_3/\bar{J}_2^{3/2}$$

where θ is the angle of similarity.

The invariants in eqns (2) are the normalized first invariant of the stress tensor

$$\bar{I}_1 = \bar{\sigma}_1 + \bar{\sigma}_2 + \bar{\sigma}_3 = \bar{\sigma}_x + \bar{\sigma}_y + \bar{\sigma}_z$$

the normalized second invariant of the stress deviator tensor

$$\begin{aligned}\bar{J}_2 &= [(\bar{\sigma}_1 - \bar{\sigma}_2)^2 + (\bar{\sigma}_2 - \bar{\sigma}_3)^2 + (\bar{\sigma}_3 - \bar{\sigma}_1)^2]/6 \\ &= [(\bar{\sigma}_x - \bar{\sigma}_y)^2 + (\bar{\sigma}_y - \bar{\sigma}_z)^2 + (\bar{\sigma}_z - \bar{\sigma}_x)^2]/6 + \bar{\tau}_{xy}^2 + \bar{\tau}_{yz}^2 + \bar{\tau}_{zx}^2\end{aligned}$$

and the normalized third invariant of the stress deviator tensor (3)

$$\begin{aligned}\bar{J}_3 &= (\bar{\sigma}_1 - \bar{\sigma}_m)(\bar{\sigma}_2 - \bar{\sigma}_m)(\bar{\sigma}_3 - \bar{\sigma}_m) \\ &= \begin{vmatrix} \bar{\sigma}_x - \bar{\sigma}_m & \bar{\tau}_{xy} & \bar{\tau}_{xz} \\ \bar{\tau}_{yx} & \bar{\sigma}_y - \bar{\sigma}_m & \bar{\tau}_{yz} \\ \bar{\tau}_{zx} & \bar{\tau}_{zy} & \bar{\sigma}_z - \bar{\sigma}_m \end{vmatrix}.\end{aligned}$$

In eqns (3) $\bar{\sigma}_1$, $\bar{\sigma}_2$, $\bar{\sigma}_3$; $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\sigma}_z$, $\bar{\tau}_{xy}$, $\bar{\tau}_{yz}$ and $\bar{\tau}_{zx}$ are normalized stress components given by

$$\bar{\sigma}_1 = \sigma_1/f_c', \qquad \bar{\sigma}_2 = \sigma_2/f_c', \ldots, \qquad \bar{\tau}_{zx} = \tau_{zx}/f_c' \tag{4}$$

in which f_c' is the uniaxial compressive strength of the concrete, σ_1, σ_2, σ_3 are the first, second and third principle stresses, respectively, and σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx} are components of the stress tensor.

In eqns (1) a_t, b_t, c_t, d_t, a_c, b_c, c_c and d_c are experimentally obtained material parameters.

The failure surface obtained from eqns (1) is shown in three-dimensional stress space in Fig. 1. It can be conveniently described by its cross-sectional axis $\bar{\sigma}_1 = \bar{\sigma}_2 = \bar{\sigma}_3$, with $\bar{\sigma}_m$ constant and its meridians in the meridional planes, which contain the hydrostatic axis, with θ constant.

If concrete is in a stress state such that

$$\bar{\sigma}_1 > \bar{\sigma}_2 = \bar{\sigma}_3 \tag{5}$$

i.e. a hydrostatic stress state with a tensile stress superimposed in one direction, then substitution of eqn (5) into eqns (2) results in $\theta = 0$. Thus the tensile meridian is given by

$$[\bar{r}]_{\theta=0} = \bar{r}_t. \tag{6}$$

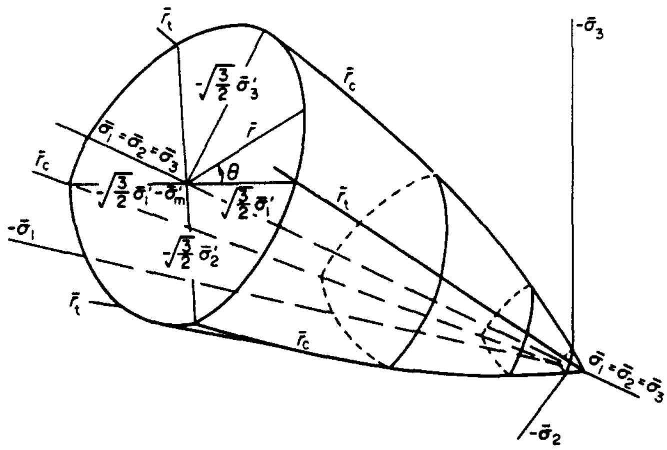

Fig. 1. Proposed three-dimensional failure surface.

For concrete in a state of stress such that

$$\bar{\sigma}_1 = \bar{\sigma}_2 > \bar{\sigma}_3 \tag{7}$$

i.e. a hydrostatic stress state with a compressive stress superimposed in one direction, $\theta = \pi/3$, the compressive meridian is given by

$$[\bar{r}]_{\theta=\pi/3} = \bar{r}_c. \tag{8}$$

Equations (6) and (8) are the geometric representations of $\bar{r}_t$ and $\bar{r}_c$, respectively.

For the case when $\bar{\sigma}_m$ is a constant then $\bar{r}_t = \bar{r}_t(\bar{\sigma}_m)$ is a constant and eqns (1) present the cross-section of the failure surface (failure curve) in a deviatoric plane. Since both $\bar{r}_t$ and n are functions of the mean normal stress, the failure curve changes along the hydrostatic axis both in shape and dimension. For $n = 1/2$

$$\bar{r} = \bar{r}_t/\cos\theta$$

and the cross-section is triangular. For $n = 1$

$$\bar{r} = \bar{r}_t$$

and the cross-section is circular. The cross-sections of the proposed failure criterion are shown in Fig. 2.

It can be shown that for $1/2 \leqslant n \leqslant 1$

$$R^2 + 2\left(\frac{\partial R}{\partial \theta}\right)^2 - R\frac{\partial^2 R}{\partial \theta^2} \geqslant 0$$

which implies that the failure curves are convex. The proof of this convexity is given in the Appendix.

DETERMINATION OF THE MATERIAL PARAMETERS

The eight material parameters a_t, b_t, c_t, d_t, a_c, b_c, c_c and d_c in the failure criterion given by eqns (1) can be determined from experiments in which concrete specimens are subjected to the following stress states:

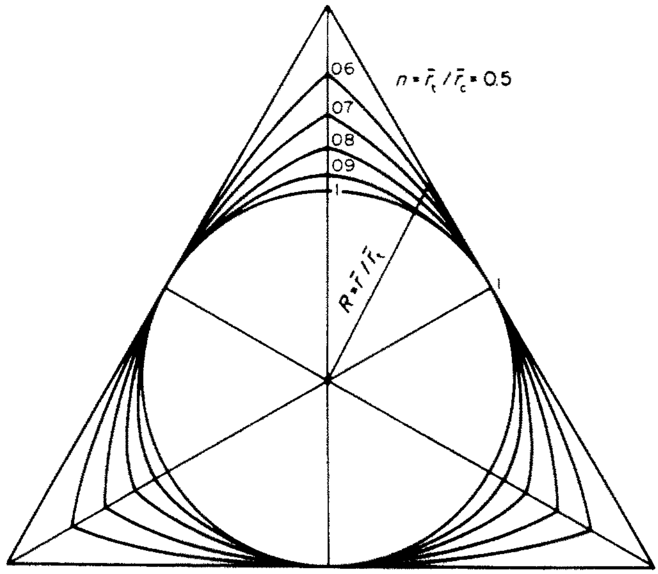

Fig. 2. Cross-sections of the proposed failure criterion.

(a) Uniaxial compressive strength f_c'

$$(\bar{\sigma}_m = -1/3, \bar{r} = \sqrt{(2/3)}, \theta = \pi/3).$$

(b) Uniaxial tensile strength f_t' in terms of compressive strength

$$\bar{f}_t' = f_t'/f_c'$$

$$(\bar{\sigma}_m = \bar{f}_t'/3, \bar{r} = \sqrt{(2/3)}\bar{f}_t', \theta = 0).$$

(c) The equal biaxial compressive strength f_{bc}', in terms of compressive strength $\bar{f}_{bc}' = f_{bc}'/f_c'$

$$\left(\bar{\sigma}_m = -\frac{2}{3}\bar{f}_{bc}', \bar{r} = \sqrt{(2/3)}\bar{f}_{bc}', \theta = 0\right).$$

(d) The triaxial compressive experimental data on the tensile meridian

$$(\bar{\sigma}_1 > \bar{\sigma}_2 = \bar{\sigma}_3), \qquad \sigma_m = -\sigma_{mt}',$$

$$r = r_t'(\bar{\sigma}_m = \bar{\sigma}_{mt}' = \sigma_{mt}'/f_c', \bar{r} = \bar{r}_t' = r_t'/f_c', \theta = 0).$$

(e) The triaxial compressive experimental data on the compressive meridian

$$(\bar{\sigma}_1 = \bar{\sigma}_2 > \bar{\sigma}_3), \sigma_m = -\sigma_{mc}',$$

$$r = r_c'(\bar{\sigma}_m = -\bar{\sigma}_{mc}' = -\sigma_{mc}'/f_c', \bar{r} = \bar{r}_c' = \bar{r}_c/f_c', \theta = \pi/3).$$

(f) The two meridians must pass through a common apex $\bar{\sigma}_{mo}'$. This requirement reduces the number of independent material parameters from eight to seven. Failure must coincide with the maximum tensile criterion for the tensile stress region given by

$$\bar{\sigma}_{mo}' = \bar{f}_t'$$

for which

$$\bar{r}_t(\bar{\sigma}'_{mo}) = \bar{r}_c(\bar{\sigma}'_{mo}) = 0$$

(g) For surface convexity, the ratio $\bar{r}_t/\bar{r}_c$ must not exceed one half. This requirement is satisfied by

$$\lim_{\bar{\sigma}_m \to \bar{f}'_t} \bar{r}_t/\bar{r}_c = 1/2.$$

Thus, there are sufficient equations for the determination of the seven independent parameters. The solution of these simultaneous equations yields

$$a_t = d_t^2 - \bar{f}'_t b_t - \bar{f}'^2_t c_t$$

$$b_t = -\left\{6\bar{f}'_{bc}(\bar{f}'_{bc} - \bar{f}'_t)\left(\bar{\sigma}'^2_{mt} + \frac{4}{3}\sqrt{\left(\frac{2}{3}\right)}\bar{r}'_t\bar{f}'_t - \bar{f}'^2_t\right) - \bar{r}'_t\left(\bar{r}'_t - \sqrt{\left(\frac{2}{3}\right)}\bar{f}'_t\right)\left(4\bar{f}'^2_{bc} + 8\bar{f}'_{bc}\bar{f}'_t - 9\bar{f}'^2_t\right)\right\} \Bigg/ \left\{9\bar{f}'_t\left(\bar{\sigma}'^2_{mt} + \frac{4}{3}\sqrt{\left(\frac{2}{3}\right)}\bar{r}'_t\bar{f}'_t - \bar{f}'^2_t\right) - \left(\bar{\sigma}'_{mt} + \bar{f}'_t - \sqrt{\left(\frac{2}{3}\right)}\bar{r}'_t\right)(4\bar{f}'^2_{bc} + 8\bar{f}'_{bc}\bar{f}'_t - 9\bar{f}'^2_t)\right\}$$

$$c_t = \{6\bar{f}'_{bc}(\bar{f}'_{bc} - \bar{f}'_t) + 9\bar{f}'_t b_t\}/\{4\bar{f}'^2_{bc} + 8\bar{f}'_{bc}\bar{f}'_t - 9\bar{f}'^2_t\} \tag{9}$$

$$d_t = -\left(\bar{f}'_t + b_t + \frac{4}{3}\bar{f}'_t c_t\right) \Big/ \sqrt{6}$$

and

$$a_c = d_c^2 - \bar{f}'_t b_c - \bar{f}'^2_t c_c$$

$$b_c = -\left\{2(\bar{\sigma}'_{mc} - \bar{f}'^2_t - 4k\bar{r}'_c\bar{f}'_t) - \bar{r}'^2_c\left(\frac{1}{3} - 3\bar{f}'^2_t - 12\sqrt{\left(\frac{2}{3}\right)}k\bar{f}'_t\right)\right\} \Bigg/ \left\{\left(1 + 3\bar{f}'_t + 6\sqrt{\left(\frac{2}{3}\right)}k\right)(\bar{\sigma}'^2_{mc} - \bar{f}'^2_t - 4k\bar{r}'_c\bar{f}'_t) - (\bar{\sigma}'_{mc} + \bar{f}'_t + 2k\bar{r}'_c)\left(\frac{1}{3} - 3\bar{f}'^2_t - 12\sqrt{\left(\frac{2}{3}\right)}k\bar{f}'_t\right)\right\}$$

$$c_c = \left\{2 + \left(1 + 3\bar{f}'_t + 6\sqrt{\left(\frac{2}{3}\right)}k\right)b_c\right\} \Bigg/ \left\{\frac{1}{3} - 3\bar{f}'^2_t - 12\sqrt{\left(\frac{2}{3}\right)}k\bar{f}'_t\right\} \tag{10}$$

$$d_c = k(b_c + 2\bar{f}'_t c_c)$$

in which

$$k = d_t/\{2(b_t + 2\bar{f}'_t c_t)\}.$$

If the assumption is made such that

$$\lim_{\bar{\sigma}_m \to -\infty} \bar{r}_t/\bar{r}_c = 1 \tag{11}$$

a_c, b_c, c_c and d_c become

$$\begin{aligned} a_c &= d_c^2 - \bar{f}'_t b_c - \bar{f}'^2_t c_c \\ b_c &= \left\{\left(\frac{1}{9} - 4\sqrt{\left(\frac{2}{3}\right)} k\bar{f}'_t - \bar{f}'^2_t\right)c_c - \frac{2}{3}\right\} \Big/ \left\{2\sqrt{\left(\frac{2}{3}\right)}k + \bar{f}'_t + \frac{1}{3}\right\} \\ c_c &= c_t \\ d_c &= k(b_c + 2\bar{f}'_t c_c) \end{aligned} \tag{10a}$$

and k has the same value as that in eqns (10).

The tensile meridian given by

$$(\bar{r}_t + d_t)^2 = a_t + b_t\bar{\sigma}_m + c_t\bar{\sigma}_m^2 \tag{12}$$

is a quadratic curve in the tensile meridian plane and to be consistent with experimental evidence it must not intersect the hydrostatic axis for high compressive stresses. For this reason the condition

$$3\bar{f}'_t\bar{r}'_t\left(\bar{r}'_t - \sqrt{\left(\frac{2}{3}\right)}\bar{f}'_t\right) - 2\bar{f}'_{bc}(\bar{f}'_{bc} - \bar{f}'_t)\left(\bar{\sigma}'_{mt} + \bar{f}'_t - \sqrt{\left(\frac{2}{3}\right)}\bar{r}'_t\right) > 0 \tag{13}$$

has to be satisfied. In this case, the tensile meridian is hyperbolic and it is convex for all $\bar{\sigma}_m$ values.

The proposed failure criterion may be reduced to earlier and simpler material failure models. The Rankine maximum tensile stress model, the von Mises model and the Drucker–Prager model are all special cases of the proposed criterion. For instance, when the parameters in eqns (1) are specified as

$$a_t = \frac{3}{2}\bar{f}'^2_t, \qquad b_t = -3\bar{f}'_t, \qquad c_t = \frac{3}{2}, \qquad d_t = 0$$

and (14)

$$a_c = 6\bar{f}'^2_t, \qquad b_c = -12\bar{f}'_t, \qquad c_c = 6, \qquad d_c = 0$$

the failure criterion becomes the maximum tensile model.

For the case when the parameters are specified as

$$a_t = a_c = \frac{2}{3}$$

and (15)

$$b_t = c_t = d_t = b_c = c_c = d_c = 0$$

the failure criterion becomes the von Mises model.

Finally, when the parameters are specified by

$$\begin{aligned} a_t = a_c &= 2\left(\frac{k'}{f'_c}\right)^2 \\ b_t = b_c &= -12\alpha\frac{k'}{f'_c} \\ c_t = c_c &= 18\alpha^2 \end{aligned} \tag{16}$$

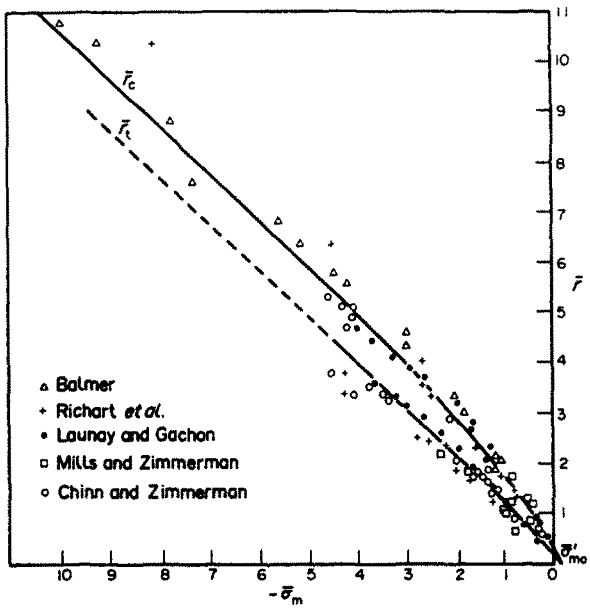

Fig. 3. Comparison of proposed criterion with experimental data.

and

$$d_t = d_c = 0$$

the failure criterion becomes the Drucker–Prager model[21]

$$\alpha I_1 + \sqrt{J_2} = k' \tag{17}$$

where α and k' are constants.

The criterion may also be reduced to represent other material failure models. For example, when the equation

$$9\bar{f}'_t\left(\bar{\sigma}'^2_{mt} + \frac{4}{3}\sqrt{\left(\frac{2}{3}\right)}\bar{r}'_t\bar{f}'_t - \bar{f}'^2_t\right) - \left(\bar{\sigma}'_{mt} + \bar{f}'_t - \sqrt{\left(\frac{2}{3}\right)}\bar{r}'_t\right)(4\bar{f}'^2_{bc} + 8\bar{f}'_{bc}\bar{f}'_t - 9\bar{f}'^2_t) = 0$$

is satisfied and

$$a_t = a_c, \qquad b_t = b_c, \qquad c_t = c_c, \qquad d_t = d_c$$

the failure criterion becomes the Bresler–Pister model[18]

$$\bar{r} = \bar{r}_t = \bar{r}_c = a + b\bar{\sigma}_m + c\bar{\sigma}^2_m$$

where a, b and c are all experimentally determined material parameters.

EXPERIMENTAL VERIFICATION

Figure 3 shows a comparison between meridians of the proposed failure surface (for $\theta = 0$ and $\pi/3$) and the experimental data reported by Balmer[19], Richart *et al.*[20], Launay and Gachon[5], Mills and Zimmerman[3] and Chinn and Zimmerman[1]. The meridians are estimated by using eqns (9) and (10a) with parameters $\bar{f}'_t = 0.1$, $\bar{f}'_{bc} = 1.15$, $\bar{\sigma}'_{mt} = 4$ and $\bar{r}'_t = 4$. The corresponding stress state on the compressive meridian is

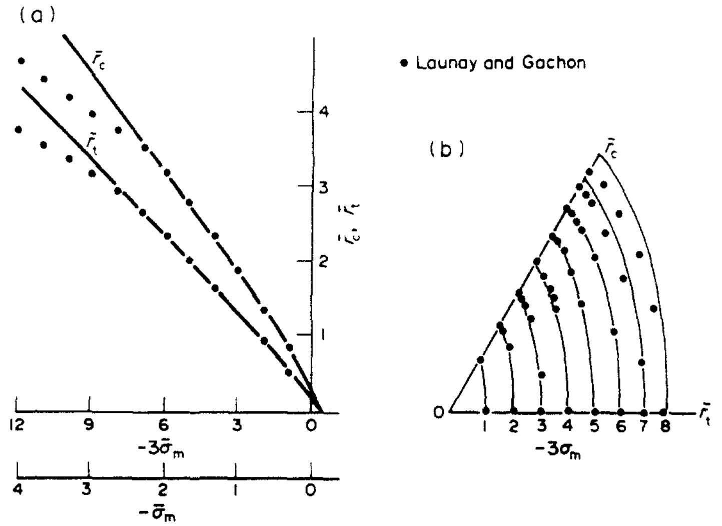

Fig. 4. Comparison of proposed criterion with the experimental results of Launay and Gachon.

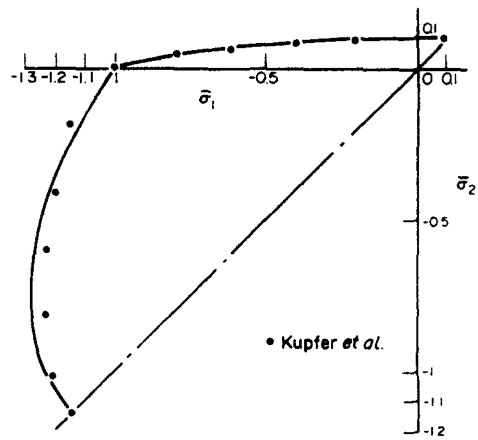

Fig. 5. Comparison of proposed criterion with the experimental results of Kupfer *et al.*

$\bar{\sigma}_{mc} = 4$ and $\bar{r}_c' = 4.92$.

For verification of the failure curves in the deviatoric planes, a comparison with the experimental data reported by Launay and Gachon[5] is shown in Fig. 4. For this case the parameters are specified as $\bar{f}_t' = 0.1$, $\bar{f}_{bc}' = 1.8$, $\bar{\sigma}_{mt}' = 2$, $\bar{r}_t' = 2.3$, $\bar{\sigma}_{mc}' = 2$ and $\bar{r}_c' = 3.15$.

The ability of the proposed failure criterion to predict the experimental biaxial results obtained by Kupfer *et al.* is shown in Fig. 5. The proposed failure curve has been obtained using the same parameters as those of Fig. 3.

CONCLUSIONS

A general, three-dimensional failure criterion for plain concrete is proposed and its applications to predict available experimental data has been demonstrated. The criterion has been developed as an alternative model to those proposed previously and in particular to the models proposed by Willam and Warnke[10]. These latter and the proposed model may be applied to the same type of problem.

The criterion includes the following features, which correspond to experimentally observed behaviour:

(a) It depends upon all stress invariants in the form of $F(\bar{\sigma}_m, \bar{r}, \theta)$.
(b) The meridians are hyperbolic and are convex everywhere.

(c) It has gradually changing cross-sections in the deviatoric planes from triangular at the apex of the failure surface, to circular at very high compressive stress state with increasing hydrostatic pressure. The cross-sections are convex everywhere.

(d) The failure surface is smooth with continuous derivative except on the compressive meridian, $\theta = \pi/3$. The fact that corners exist in the proposed formulation causes certain practical difficulties during incremental elastic–plastic analysis but these can, however, be overcome[22].

(e) The failure surface intersects the hydrostatic axis only at the apex.

(f) It coincides with the maximum tensile criterion for the tensile stress region.

(g) The maximum tensile, the von Mises, the Drucker–Prager and the Bresler–Pister models are all special cases of the proposed criterion.

REFERENCES

1. J. Chinn and R. M. Zimmerman, Behaviour of plain concrete under various high triaxial compressive loading conditions. Air Force Weapons Lab. Tech. Rep. WL TR 64-163 (August 1965).
2. H. Kupfer, H. K. Hilsdrof and H. Rusch, Behaviour of concrete under biaxial stresses. *J. Am. Concr. Inst.* **66**(8), 656–666 (August 1969).
3. L. L. Mills and R. M. Zimmerman, Compressive strength of plain concrete under multiaxial loading conditions. *J. Am. Concr. Inst.* **67**(10), 802–807 (October 1970).
4. K. Newman and J. B. Newman, Failure theories and design criteria for plain concrete. *Structure, Solid Mechanics and Engineering Design* (Edited by M. Te'eni), pp. 963–995. Wiley-Interscience, London (1971).
5. P. Launay and H. Gachon, Strain and ultimate strength of concrete under triaxial stress. *Am. Concr. Inst. Spec. Publ.* **34**, paper 13 (1972).
6. T. C. Y. Liu, A. H. Nelson and F. O. Slate, Stress–strain response and fracture of concrete in uniaxial and biaxial compression. *J. Am. Concr. Inst.* **69**, 291–295 (May 1972).
7. F. Bremer, On a triaxial strength criterion for concrete, *Am. Concr. Inst. Spec. Publ.* **34**, paper 14 (1972).
8. V. Hansson and K. Schimmelpfennig, Concrete strength in multiaxial stress state. *Am. Concr. Inst. Spec. Publ.* **34**, paper 15 (1972).
9. S. S. Hsieh, E. C. Tring and W. F. Chen, An elastic–fracture model for concrete. *Proc. 3rd Engng Mech. Div. Spec. Conf. ASCE*, Austin Texas, pp. 437–440 (1979).
10. K. J. Willam and E. P. Warnke, Constitutive models for the triaxial behaviour of concrete. Int. Ass. Bridge Struct. Eng. Sem. Concr. Struct. Subjected Triaxial Stresses, Bergamo, Italy (1974).
11. N. S. Ottosen, A failure criterion for concrete. *J. Engng Mech. Div. ASCE* **103**(EM 4), 527–535 (August 1977).
12. K. H. Gerstle *et al.*, Behaviour of concrete under multiaxial stress states. *J. Engng Mech. Div. ASCE* **106**(EM 6), 1383–1403 (December 1980).
13. E. Y. T. Chen and W. C. Schnobrich, Material modelling of plain concrete. Int. Ass. Bridge Struct. Eng., Colloquium on Advanced Mechanics of Reinforced Concrete, Delft, pp. 33–51 (1981).
14. P. V. Lade, Three-parameters failure criterion for concrete. *J. Engng Mech. Div. ASCE* **108**(EM 5), 850–863 (October 1982).
15. W. F. Chen, *Plasticity in Reinforced Concrete*, pp. 190–250. McGraw-Hill, New York (1982).
16. J. Wastiels, Behaviour of concrete under multiaxial stress states—a review. *Cement and Concrete Research*, Vol. 9, pp. 35–44. Pergamon Press, New York (1979).
17. J. H. Argyris, G. Faust, J. Szimmat, E. P. Warnke and K. J. Willam, Recent developments in the finite element analysis of prestressed concrete reactor vessels. *Nucl. Engng Des.* **28**, 42–75 (1974).
18. B. Bresler and K. S. Pister, Strength of concrete under combined stresses. *J. Am. Concr. Inst.* **55**, 321–345 (1958).
19. G. G. Balmer, Shearing strength of concrete under high triaxial stress-computation of Mohr's envelope as a curve. Bur. Reclam. Struct. Res. Lab. Rep. SP-23 (1949).
20. F. E. Richart, A. Brandtzaeg and R. L. Brown, A study of the failure of concrete under combined compressive stresses. *Univ. Ill. Engng Exp. Stn Bull.* **185** (1928).
21. D.C. Drucker and W. Prager, Soil mechanics and plastic analysis or limit design. *Q. Appl. Math.* **10**(2), 157–165 (1952).
22. G. C. Nayak and O. C. Zienkiewicz, Elasto-plastic stress analysis. A generalization for various constitutive relations including strain softening. *Int. J. Numer. Meth. Engng* **5**, 113–135 (1972).

APPENDIX: CONVEXITY OF THE FUNCTION $R = \bar{r}/\bar{r}_t = \{3/[3 - 4(1 - n^2)\sin^2\theta]\}^{1/2}$

Since

$$R^2 = 3/[3 - 4(1 - n^2)\sin^2\theta]$$

$$2R\frac{\partial R}{\partial \theta} = \frac{4}{3}(1 - n^2)R^4 \sin 2\theta$$

and

$$\frac{\partial R}{\partial \theta} = \frac{2}{3}(1 - n^2)R^3 \sin 2\theta$$

$$\frac{\partial^2 R}{\partial \theta^2} = \frac{4}{3}(1 - n^2)R^3[(1 - n^2)R^2 \sin^2 2\theta + \cos 2\theta]$$

$$R^2 + 2\left(\frac{\partial R}{\partial \theta}\right)^2 - R\frac{\partial^2 R}{\partial \theta^2} = R^2\left\{1 - \frac{4}{3}(1 - n^2)R^2\left[\frac{1}{3}(1 - n^2)R^2 \sin^2 2\theta + \cos 2\theta\right]\right\}.$$

Since $1/2 \leqslant n \leqslant 1$, $0 \leqslant (1 - n^2) \leqslant 3/4$ and noting $R \leqslant 1/\cos\theta$ then

$$\begin{aligned}
&\frac{4}{3}(1 - n^2)R^2\left[\frac{1}{3}(1 - n^2)R^2 \sin^2 2\theta + \cos 2\theta\right] \\
&\leqslant \left[\frac{\sin^2 2\theta}{4\cos^2\theta} + \cos 2\theta\right]\Big/\cos^2\theta \\
&= \cos^2\theta/\cos^2\theta \\
&= 1
\end{aligned}$$

Thus

$$R^2 + 2\left(\frac{\partial R}{\partial \theta}\right)^2 - R\frac{\partial^2 R}{\partial \theta^2} \geqslant 0$$

and therefore, the failure surface given by

$$R = \bar{r}/\bar{r}_t = \{3/[3 - 4(1 - n^2)\sin^2\theta]\}$$

is convex everywhere for $1/2 < n < 1$.